Chiral density waves in quark matter within the Nambu–Jona-Lasinio model in an external magnetic field

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A possibility of formation of static dual scalar and pseudoscalar density wave condensates in dense quark matter is considered for the Nambu–Jona-Lasinio model in an external magnetic field. Within a mean-field approximation, the effective potential of the theory is obtained and its minima are numerically studied; a phase diagram of the system is constructed. It is shown that the presence of a magnetic field favors the formation of spatially inhomogeneous condensate configurations at low temperatures and arbitrary nonzero values of the chemical potential.

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I. INTRODUCTION

At present, one of the most commonly used effective theories of quantum chromodynamics is the Nambu–Jona-Lasinio (NJL) model [1, 2], a local relativistic four-fermion interaction theory. The QCD and NJL Lagrangians possess the same symmetry group, the NJL model is therefore widely exploited in studying the nonperturbative QCD vacuum and its properties under various external conditions. Many features of quarks and light mesons can be successfully described within the NJL model on the basis of the spontaneous chiral symmetry breaking phenomenon [3–5].

Considering the QCD ground state properties, a number of studies were dedicated to the possibility of formation of spatially nonuniform phases in dense quark matter. It was first shown [6] that spatially inhomogeneous and anisotropic chiral condensation may occur in QCD at asymptotically high values of the chemical potential and large N_c, the ground state spatial structure taking the form of a standing wave. This phenomenon was further discussed in literature [7–9] investigating the possibility of such type of symmetry breaking and its competition with color superconductivity under various conditions including intermediate densities of quark matter. The problem has also been examined recently in the context of quarkyonic matter [10]. Along with QCD studies, similar behavior of the ground state has also been discovered and successfully reproduced in NJL-like effective models [9, 11–27], although the effect tends to be dependent on the adopted regularization scheme (see, e.g., Ref. [28] for details).

Spatially nonuniform condensates proposed at the start of the theoretical research on the subject [11] and studied extensively later on are known as dual chiral density waves (DCDW, the name introduced in Ref. [20]). The corresponding configuration can be described as follows:

$$\begin{aligned} \langle\overline{\psi}\psi\rangle &= \Delta\cos\mathbf{qr}, \\ \langle\overline{\psi}i\gamma^5\tau_3\psi\rangle &= \Delta\sin\mathbf{qr}, \end{aligned} \tag{1}$$

where Δ is the chiral density amplitude, $\mathbf{q}$ is a wave vector (which has to be determined dynamically along with Δ), and τ_a are the isospin Pauli matrices. Expectation values $\langle\overline{\psi}\psi\rangle$ and $\langle\overline{\psi}i\gamma^5\tau_3\psi\rangle$ are identified with σ and π^0 condensates; one generally assumes $\langle\overline{\psi}i\gamma^5\tau_1\psi\rangle = \langle\overline{\psi}i\gamma^5\tau_2\psi\rangle = 0$, thus charged $\pi^\pm$ condensates being absent. In general, scalar and pseudoscalar condensates are on the chiral circle: $\langle\overline{\psi}\psi\rangle^2 + \langle\overline{\psi}i\gamma^5\boldsymbol{\tau}\psi\rangle^2 = \Delta^2$. It is argued (see, e.g., Refs. [19, 20]) that DCDW may arise between the massive and symmetric phases of the NJL model at low temperatures if the coupling constant is sufficiently large. The formation of DCDW along with color superconductivity has also been discussed in literature [21–23]. It should be noted however that, although the majority of studies of condensate inhomogeneity

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focus on wavelike configurations and DCDW in particular, this is mainly for technical reasons. There may exist other competing and even more preferable spatially nonuniform ground state configurations like domain walls, see, e.g., Ref. [29], but, in general, they are much harder to deal with. For the same technical reasons, one usually considers the limit of vanishing quark current masses (chiral limit), although recently efforts to get rid of this assumption have been made [24–26].

The chiral condensation phenomenon (with spatially homogeneous condensate configurations) has recently attracted great attention in the situation when external gauge fields and, in particular, strong magnetic fields are present [30–40]. In fact these fields are common in the physical circumstances where the phase structure of quark matter is of interest, e.g., in compact stars or in heavy-ion collision processes [41]. The effect of magnetic fields on the spatially nonuniform chiral condensation is therefore worth investigation. This is important in the context of a targeted experimental search for possible condensate inhomogeneity signals predicted, e.g., in Ref. [19]. Some interesting results related to the subject have been obtained in Ref. [42] showing that a stack of π^0 domain walls may emerge in the QCD vacuum in a strong magnetic field due to the axial anomaly (and concerning color superconductivity in a magnetic field, see, e.g., Refs. [43, 44] suggesting the formation of magnetic domains in cores of compact stars).

In this paper, we examine the formation of the condensate configuration defined in Eq. (1) in dense quark matter in the framework of the NJL model in the presence of an external magnetic field and show that the latter favors the emergence of DCDW at low temperatures. Limiting ourselves to the chiral limit, we base our calculations upon exact solutions of the Dirac equation and use the proper-time regularization method such that our results agree with Ref. [20] in the zero-field limit.

II. THE MODEL

We start from the NJL Lagrangian density for a quark field ψ with $N_f = 2$ flavors (representing the up- and down-quarks) and $N_c = 3$ colors:

$$\mathcal{L}^{\mathrm{NJL}} = \overline{\psi}\left(i\gamma D - m_c + \mu\gamma^0\right)\psi + G\left[(\overline{\psi}\psi)^2 + (\overline{\psi}i\gamma^5\boldsymbol{\tau}\psi)^2\right], \tag{2}$$

where G is the coupling constant, μ is the chemical potential, m_c is the quark current mass, the covariant derivative $D = \partial + iQA$ with A being the electromagnetic field, and Q the electric charge matrix acting in the flavor space:

$$Q = \begin{pmatrix} \frac{2}{3}e & 0 \\ 0 & -\frac{1}{3}e \end{pmatrix}, \quad e > 0.$$

We take $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ and assume τ_3 to be diagonal, we use the standard (Dirac) representation of the γ matrices throughout the paper; matrix indices are suppressed in our notation when possible. In what follows, we hold $m_c = 0$ assuming that the appropriate dimensional parameters in our model tend to be much greater than $m_c \simeq 5\,\mathrm{MeV}$. The symmetry of the model is therefore $SU_L(2) \times SU_R(2)$ and it is reduced to $U_{\tau_3 L}(1) \times U_{\tau_3 R}(1)$ when an external homogeneous magnetic field is present (with the field strength pointing in the z direction).

Using ansatz (1), we obtain the Lagrangian density in the mean-field approximation (we only take into account Hartree terms here, see a discussion on this subject in Ref. [20]):

$$\mathcal{L}^{\mathrm{MF}} = \overline{\psi}\left[i\gamma D + \mu\gamma^0 - m\left(\cos\mathbf{qr} + i\gamma^5\tau_3\sin\mathbf{qr}\right)\right]\psi - \frac{m^2}{4G}, \tag{3}$$

where we have denoted $m = -2G\Delta$. We assume that the system resides in an external magnetic field, the wave vector $\mathbf{q}$ being parallel to the field strength $\mathbf{H}$, both vectors oriented along the z axis. Such an assumption is reasonable due to the symmetry considerations; possible small deviations of $\mathbf{q}$ from the preferred orientation along $\mathbf{H}$ are taken into account further.

As it is commonly done when considering model (3), we use a field transformation $\psi \to e^{i\gamma^5\tau_3 bx}\,\psi$, $\overline{\psi} \to \overline{\psi}\,e^{i\gamma^5\tau_3 bx}$, where $b^\mu \equiv (0, \mathbf{b})$, $x^\mu \equiv (t, \mathbf{r})$, and $\mathbf{b} = \mathbf{q}/2$, to remove the spatial modulation from the resulting Lagrangian density $\mathcal{L}$:

$$\mathcal{L} = \overline{\psi}\left(i\gamma D + \mu\gamma^0 - m + \gamma^5\tau_3\gamma b\right)\psi - \frac{m^2}{4G}. \tag{4}$$

It should be noted, however, that special care is needed when performing such operations in the presence of background gauge fields. To obtain correct results, one should apply, for example, Fujikawa's method [45] and its generalizations for finite fermion field transformations. Fortunately, the path integral measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ remains invariant in our case since the quantity $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ which arises in Fujikawa's exponent, where F is the electromagnetic field strength and ϵ is the antisymmetric tensor, vanishes in the absence of an electric field.

In what follows, we obtain the thermodynamic potential Ω for the model described by Eq. (4) and then study numerically the minima of Ω with respect to the order parameters m and $\mathbf{b}$.

III. ONE-PARTICLE ENERGY SPECTRUM

For later convenience, let us first consider a simplified model for a charged fermion (electron) field having no flavors or colors with the Lagrangian density

$$\mathcal{L} = \overline{\psi}\left(i\gamma D - m - \gamma^5\gamma b\right)\psi, \tag{5}$$

where $D = \partial - ieA$, $e > 0$. The term $\overline{\psi}\gamma^5\gamma b\psi$ in the latter expression describes a Lorentz- and CPT-breaking background interaction controlled by the axial four-vector b^μ. This type of interaction arising within the context of the Standard Model Extension [46] has been a subject of extensive theoretical research in recent years (see, e.g., Refs. [47–50]). In this paper, in order to obtain the one-particle energy spectrum of model (5), we use a technique similar to that adopted in Ref. [50].

Let $\mathbf{b} = (0, 0, b)$, $\mathbf{H} = (0, 0, H)$, $H > 0$; we take the electromagnetic field in the Landau gauge: $A^\mu = (0, \mathbf{A})$, $\mathbf{A} = (0, Hx, 0)$. The modified Dirac Hamiltonian derived from Eq. (5) is as follows:

$$H_{\mathrm{D}} = \boldsymbol{\alpha}\mathbf{P} + \gamma^0 m - \Sigma_3 b, \tag{6}$$

where $\mathbf{P} = -i\nabla + e\mathbf{A}$ is the gauge-invariant kinetic momentum, $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, $\Sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma^{jk}$, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Since $[(\boldsymbol{\alpha}\mathbf{P}_\perp)^2, H_{\mathrm{D}}] = 0$ where $\mathbf{P}_\perp = (P_1, P_2, 0)$, $(\boldsymbol{\alpha}\mathbf{P}_\perp)^2$ being an observable with an oscillatorlike spectrum, it is easy to prove that the eigenfunctions of H_{D} have a standard general form (see Chapter IV of Ref. [51] for details):

$$\Psi_{nqp}(x, y, z) = \frac{1}{\sqrt{2\pi}}\, e^{ipz}\, \frac{1}{\sqrt{2\pi}}\, e^{iqy} \begin{pmatrix} c_1\, u_{n-1}(\xi) \\ ic_2\, u_n(\xi) \\ c_3\, u_{n-1}(\xi) \\ ic_4\, u_n(\xi) \end{pmatrix} (eH)^{1/4}, \quad \xi = \sqrt{eH}x + \frac{q}{\sqrt{eH}}, \tag{7}$$

where $u_n(\xi)$ are the orthonormalized Hermite functions [we assume $u_{-1}(\xi) \equiv 0$] and $\{c_i\}$ are spin-dependent coefficients. The quantum number $n = 0, 1, \ldots$ is the Landau level, p is the momentum component parallel to the magnetic field direction, and q is related to the symmetry center x_0 of the wavefunction Ψ along the x axis: $q = -x_0\, eH$. For each $n > 0$ and fixed q and p, we have an eigenvalue problem for a 4×4-sized matrix K acting on the vector $\{c_i\}$, where

$$K = \alpha_1 p_\perp + \alpha_3 p + \gamma^0 m - \Sigma_3 b, \quad p_\perp = \sqrt{2eHn}. \tag{8}$$

The quantity q is absent in Eq. (8) thus providing the degeneracy of the energy spectrum with respect to it; this phenomenon is related to the freedom in placing the particle's orbit in a magnetic field and is preserved for any gauge of $\mathbf{A}$.

Let us now consider a unitary transformation: $\tilde{K} = U^{-1}KU$ where $U = e^{i\Sigma_2\frac{\pi}{2}}e^{i\gamma^0\Sigma_2\frac{\pi}{2}} = \frac{1}{2}(1 + i\Sigma_2)(1 + i\gamma^0\Sigma_2)$; it yields

$$\tilde{K} = \alpha_1\tilde{p}_\perp + \alpha_3\tilde{p} + \gamma^0 m + \gamma^0\Sigma_3\tilde{\mu}H, \quad \tilde{p}_\perp = p, \quad \tilde{p} = -p_\perp, \quad \tilde{\mu}H = b. \tag{9}$$

The matrix $\tilde{K}$ formally corresponds to an electron with an effective vacuum magnetic moment moving in an effective external magnetic field. The problem for this case has been studied and solved in Ref. [52] (note that the form of the coefficients $\{c_i\}$ is independent of the adopted electromagnetic field gauge). The case $n = 0$ requires a separate treatment though, since K is reduced to a 2×2-sized matrix K_0 acting on the coefficients $\{c_i\}$, $i = 2, 4$:

$$K_0 = \begin{pmatrix} m + b & -p \\ -p & -m + b \end{pmatrix}.$$

The eigenvalue problem for K_0 can easily be solved. The final expression for the energy spectrum has the form

$$E_{np\zeta\epsilon} = \begin{cases} \epsilon\sqrt{\left(\zeta\sqrt{m^2 + p^2} + b\right)^2 + 2eHn}, & n = 1, 2, \ldots, \\ \epsilon\sqrt{m^2 + p^2} + b, & n = 0, \end{cases} \tag{10}$$

where $\zeta = \pm 1$ is the spin quantum number, $\epsilon = \pm 1$ is the energy sign (when $n > 0$). When $n = 0$, one only has two (instead of four for $n > 0$) energy branches distinguished by the number ϵ and the latter has lost its meaning of the energy sign in the presence of $b \neq 0$. Spectrum (10) has been known in literature [53] but the energy shift of the $n = 0$ level has not been shown explicitly in the paper cited. The specific asymmetry between the particle and antiparticle

energy spectra is due to the CPT-odd nature of the background interaction present in our model. The phenomenon does not manifest itself for free particles since one can compensate the CPT-induced transformation $\mathbf{b} \to -\mathbf{b}$ by a spatial rotation. But this can no longer be done in the presence of a preferred spatial direction which is introduced with $\mathbf{H}$ in our problem.

The coefficients $\{c_i\}$ which meet the orthonormalization requirement for the eigenfunctions $\{\Psi_{nqp\zeta\epsilon}\}$ are as follows:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -\zeta B(P-\epsilon\zeta Q) \\ -A(P+\epsilon\zeta Q) \\ A(P-\epsilon\zeta Q) \\ -\zeta B(P+\epsilon\zeta Q) \end{pmatrix}, \tag{11}$$

where

$$A = \sqrt{1+\frac{m}{\Pi}}, \quad B = \sqrt{1-\frac{m}{\Pi}}, \quad P = \sqrt{1-\frac{p_\perp}{E}}, \quad Q = \sqrt{1+\frac{p_\perp}{E}}; \quad \Pi = \zeta\sqrt{m^2+p^2}.$$

Formula (11) is valid for all n provided that one assumes $\zeta = \epsilon$ when $n = 0$. This has a physical reason since the quantity Π is an eigenvalue of the spin operator $\gamma^5(P_3 - \gamma^3 m)$ which commutes with H_{D}, and $\Pi = \epsilon\sqrt{m^2+p^2}$ at the lowest Landau level.

Now that we have found the energy spectrum and a system of wavefunctions for Hamiltonian (6), we may use the perturbation theory to take into account possible small deviations of $\mathbf{b}$ from the direction of the magnetic field. Let $\mathbf{b} = (b_\perp, 0, b)$, the corresponding correction to H_{D} being $V = -\Sigma_1 b_\perp$. There is no first-order correction to the energy due to the rotational symmetry of the system. The second-order correction obtained through the standard procedure is as follows:

$$\Delta E_\perp = \left(\left.\frac{R_+R'_-}{E-E'}\right|_{\substack{n'=n+1\\ \epsilon'=\epsilon}} + \left.\frac{R_+R'_-}{E-E'}\right|_{\substack{n'=n+1\\ \epsilon'=-\epsilon}} + \left.\frac{R'_+R_-}{E-E'}\right|_{\substack{n'=n-1\\ \epsilon'=\epsilon}} + \left.\frac{R'_+R_-}{E-E'}\right|_{\substack{n'=n-1\\ \epsilon'=-\epsilon}} \right)\Bigg|_{\substack{p'=p\\ \zeta'=-\zeta}}, \tag{12}$$

where E is given by Eq. (10),

$$R_\pm = \sqrt{2} b_\perp \left(1 \pm \zeta\epsilon\sqrt{1-\frac{p_\perp^2}{E^2}}\right),$$

and we have used a stroke symbol to denote that a quantity is a function of the quantum number set $\{n'p'\zeta'\epsilon'\}$ instead of $\{np\zeta\epsilon\}$, the latter being fixed for a given one-particle state. The numbers n', ϵ' are expressed through n, ϵ differently for each term in Eq. (12) while one has $p' = p$ and $\zeta' = -\zeta$ in the whole expression. The last two terms with $n' = n-1$ are absent in the case $n = 0$ due to $R_- = 0$ provided that $\zeta = \epsilon$.

Despite the emerging energy level degeneracy with respect to ζ, Eq. (12) is valid in the limit $b \to 0$. It is easy to notice though that the terms with $\epsilon' = \epsilon$ suffer from divergence due to a level crossing possible for states with adjacent n and opposite ζ, thus making the result obtained not applicable in the corresponding region of the parameter space, namely, when $4b^2\left(m^2+p^2\right) \simeq (eH)^2$ [if $\zeta = \operatorname{sgn} b$ the first term in Eq. (12) is divergent and if $\zeta = -\operatorname{sgn} b$ such is the third]. To workaround this, one has to modify the method of calculating $\Delta E_\perp$ in that region. Using the perturbation theory formalism for two near-degenerate levels with energies E, E' (see, e.g., Ref. [54] for details), we find new energy values $E_\pm$ with a gap induced by the perturbation V:

$$E_\pm = \frac{1}{2}\left(E + E' \pm \sqrt{(E-E')^2 + 4RR'}\right), \quad RR' = \begin{cases} R_+R'_-, & n' = n+1, \\ R'_+R_-, & n' = n-1. \end{cases} \tag{13}$$

Assuming $E_\pm = E + \Delta E_\perp$ and taking into account that

$$\operatorname{sgn}(E-E')\sqrt{(E-E')^2+4RR'} \simeq E - E' + \frac{2RR'}{E-E'}$$

when $RR' \ll |E-E'|$, Eqs. (12) and (13) can be combined into one asymptotic formula with the change

$$\frac{RR'}{E-E'} \to \frac{1}{2}\left(-E + E' + \operatorname{sgn}(E-E')\sqrt{(E-E')^2+4RR'}\right)$$

applied to the first and the third term in Eq. (12), with RR' being $R_+R'_-$ and R'_+R_- respectively. The factor $\operatorname{sgn}(E-E')$ is used in the latter expression to select a proper branch of solution (13) since we want to retain the

meaning of $\Delta E_\perp$ being a small correction to a particular energy level E when $|E-E'|$ is not vanishing. There is an ambiguity in this approach arising when $E=E'$ in either of the terms which have undergone the change, it may be fixed with the help of the following convention:

$$\mathrm{sgn}(E-E') \to \begin{cases} \mathrm{sgn}_+(E-E'), & n'=n+1, \\ \mathrm{sgn}_-(E-E'), & n'=n-1, \end{cases}$$

where $\mathrm{sgn}_\pm(0)=\pm 1$. It is easy to see that this ensures the consistency of the formula obtained (no values of $\Delta E_\perp$ have been lost when considering the energy spectrum as a whole). The final result reads

$$\begin{aligned} \Delta E_\perp = \Bigg[& \frac{1}{2}\left(-E+E'+\mathrm{sgn}_+(E-E')\sqrt{(E-E')^2+4R_+R'_-}\right)\Bigg|_{\substack{n'=n+1\\ \epsilon'=\epsilon}} + \frac{R_+R'_-}{E-E'}\Bigg|_{\substack{n'=n+1\\ \epsilon'=-\epsilon}} \\ & + \frac{1}{2}\left(-E+E'+\mathrm{sgn}_-(E-E')\sqrt{(E-E')^2+4R'_+R_-}\right)\Bigg|_{\substack{n'=n-1\\ \epsilon'=\epsilon}} + \frac{R'_+R_-}{E-E'}\Bigg|_{\substack{n'=n-1\\ \epsilon'=-\epsilon}}\Bigg]\Bigg|_{\substack{p'=p\\ \zeta'=-\zeta}}. \end{aligned} \tag{14}$$

The spectrum $E+\Delta E_\perp$ with E and $\Delta E_\perp$ provided with Eqs. (10) and (14) can now be used to evaluate the effective action of the model.

IV. EFFECTIVE POTENTIAL AND REGULARIZATION

Let us now return to model (4). The corresponding one-loop effective action

$$\Gamma = \int d^4x \left(-\frac{m^2}{4G}\right) + \frac{1}{i}\ln \mathrm{Det}\left(i\gamma D + \mu\gamma^0 - m + \gamma^5\tau_3\gamma b\right) \tag{15}$$

is decomposed trivially into similar parts calculated separately for each flavor and color; moreover, it can be expressed in terms of the effective action Γ' for the model studied in the previous section with an appropriate change in the electric charge and the chemical potential included:

$$\begin{aligned} \Gamma &= \int d^4x \left(-\frac{m^2}{4G}\right) + N_c\,\Gamma'\big|_{e\to\frac{2}{3}e} + N_c\,\Gamma'\big|_{e\to\frac{1}{3}e}, \\ \Gamma' &= \frac{1}{i}\ln\mathrm{Det}\left(i\gamma D+\mu\gamma^0-m-\gamma^5\gamma b\right) = \frac{1}{i}\ln\mathrm{Det}\left(i\partial^0+\mu-H_{\mathrm{D}}\right), \end{aligned} \tag{16}$$

where H_{D} is given in Eq. (6). We have used a charge conjugation for the up-quark when deriving the foregoing. Since we know the eigenfunctions $\{\Psi_{nqp\zeta\epsilon}\}$ and the spectrum $\{E_{np\zeta\epsilon}\}$ of H_{D}, the expression for Γ' can be evaluated through the standard procedure:

$$\begin{aligned} \Gamma' &= \frac{1}{2i}\mathrm{Tr}\ln\left[-(i\partial^0)^2+(H_{\mathrm{D}}-\mu)^2\right] \\ &= \frac{1}{2i}\int dp^0 \sum_{(n)} \int d^4x\, \frac{1}{\sqrt{2\pi}}e^{ip^0t}\,\Psi^+\,\ln\left[-(i\partial^0)^2+(H_{\mathrm{D}}-\mu)^2\right]\frac{1}{\sqrt{2\pi}}e^{-ip^0t}\,\Psi \\ &= \frac{1}{2i}\int dp^0 \sum_{(n)} \ln\left[-(p^0)^2+(E-\mu)^2\right]\frac{L_t}{2\pi}\frac{L_z}{2\pi}\frac{L_y}{2\pi}, \end{aligned}$$

where we have introduced a characteristic four-volume $L_tL_xL_yL_z$ and

$$\sum_{(n)} \equiv \sum_{n\zeta\epsilon}\int dp\int dq = \sum_{n\zeta\epsilon}\int dp\, eHL_x.$$

In order to obtain the thermodynamic potential Ω, we employ Matsubara's technique [55]:

$$\int\limits_{-\infty}^{\infty}\frac{dp^0}{2\pi} \to \int\limits_{-i\infty}^{i\infty}\frac{dp^0}{2\pi} = i\int\limits_{-\infty}^{\infty}\frac{dp^4}{2\pi} \to i\frac{1}{\beta}\sum_{k=-\infty}^{+\infty}, \quad p^0 \to ip^4 \to i\omega_k = i\frac{2\pi}{\beta}\left(k+\frac{1}{2}\right),$$

where $\beta = 1/T$ is the inverse temperature; the sum over k is easily evaluated. We finally find

$$\begin{aligned} \Omega &= -\frac{\Gamma}{L_t L_x L_y L_z} = \frac{m^2}{4G} + N_c\, \Omega'\big|_{e\to\frac{2}{3}e} + N_c\, \Omega'\big|_{e\to\frac{1}{3}e}, \\ \Omega' &= -\frac{1}{2}\frac{eH}{(2\pi)^2}\int dp \sum_{n\zeta\epsilon}\left[|E-\mu| + \frac{2}{\beta}\ln\left(1+e^{-\beta|E-\mu|}\right)\right]. \end{aligned} \tag{17}$$

Separating the effects of nonzero temperature and the vacuum contribution, expression (17) can be decomposed into three terms:

$$\Omega' = \Omega'_{\rm v} + \Omega'_\mu + \Omega'_T,$$

where

$$\Omega'_{\rm v} = -\frac{1}{2}\frac{eH}{(2\pi)^2}\int dp \sum_{n\zeta\epsilon}|E|, \tag{18}$$

$$\Omega'_\mu = -\frac{1}{2}\frac{eH}{(2\pi)^2}\int dp \sum_{n\zeta\epsilon}\left(|E-\mu| - |E|\right), \tag{19}$$

$$\Omega'_T = -\frac{1}{\beta}\frac{eH}{(2\pi)^2}\int dp \sum_{n\zeta\epsilon}\ln\left(1+e^{-\beta|E-\mu|}\right). \tag{20}$$

The vacuum term $\Omega'_{\rm v}$ is divergent while the terms Ω'_μ and Ω'_T are finite (being zero when $\mu = 0$ and $T = 0$ respectively). The NJL model is known to be sensitive to the choice of a regularization scheme due to the nonrenormalizable nature of the four-fermion interaction [56] (see also a discussion on this subject for the case of a spatially nonuniform condensate in Ref. [28]). We here employ the proper-time method [57]:

$$\Omega'_{\rm v} \to \frac{1}{4\sqrt{\pi}}\frac{eH}{(2\pi)^2}\int dp \sum_{n\zeta\epsilon}\int\limits_{1/\Lambda^2}^{+\infty}\frac{ds}{s^{3/2}}\, e^{-sE^2},$$

where Λ is the regularization parameter; so our results should agree with those obtained in Ref. [20] in the limit $H \to 0$.

It is easy to see that expression (20) for Ω'_T is well defined, and although expression (19) for Ω'_μ seems to be convergent due to the internal sum over ϵ, care is needed when evaluating it since it has been obtained as a difference of two divergent objects. One can derive an arbitrary value for such an expression rearranging the terms during the summation procedure [58], so an intermediate regularization is needed to get a correct result. Let it be a simple cutoff:

$$\Omega'_\mu \to -\frac{1}{2}\frac{eH}{(2\pi)^2}\int dp \sum_{n\zeta\epsilon}\left(|E-\mu| - |E|\right)\,\theta(\Lambda' - |E|), \tag{21}$$

where Λ' is sufficiently large (not necessarily being equal to Λ). If the relation $E|_{\epsilon=+1} = -E|_{\epsilon=-1} > 0$ holds, the cutoff factor can be dropped out (provided that $\mu < \Lambda'$):

$$\begin{aligned} &\sum_\epsilon \left(|E-\mu| - |E|\right)\,\theta(\Lambda' - |E|) = \left[\left(|E-\mu| + E + \mu - 2E\right)\,\theta(\Lambda' - E)\right]\big|_{\epsilon=+1} \\ &= \left[2\left(\mu - E\right)\,\theta(\mu - E)\,\theta(\Lambda' - E)\right]\big|_{\epsilon=+1} = \left[2\left(\mu - E\right)\,\theta(\mu - E)\right]\big|_{\epsilon=+1}. \end{aligned}$$

But this is not the case when the symmetry between the particle and antiparticle spectra is broken: $E|_{\epsilon=+1} \neq -E|_{\epsilon=-1}$, which occurs at the lowest Landau level in our problem [and when taking into account corrections (14) to the energy levels as well]. In general, one has to retain the regularization throughout the calculations or modify the whole expression by a finite but nonzero correction. The effect for the case $b_\perp = 0$, $n = 0$ can be studied exactly (see the Appendix):

$$\int dp \sum_\epsilon \left(|E-\mu| - |E|\right)\,\theta(\Lambda' - |E|)\bigg|_{\Lambda'\to\infty} = \int dp \sum_\epsilon \left(|E-\mu| - |E|\right) + 4\mu b.$$

If one omits the term $4\mu b$ in the above expression, the resulting potential Ω turns to be dependent on b when $m = 0$, and this is physically incorrect according to definition (1); no observable quantity may depend on the wave vector of a condensate wave with a zero amplitude.

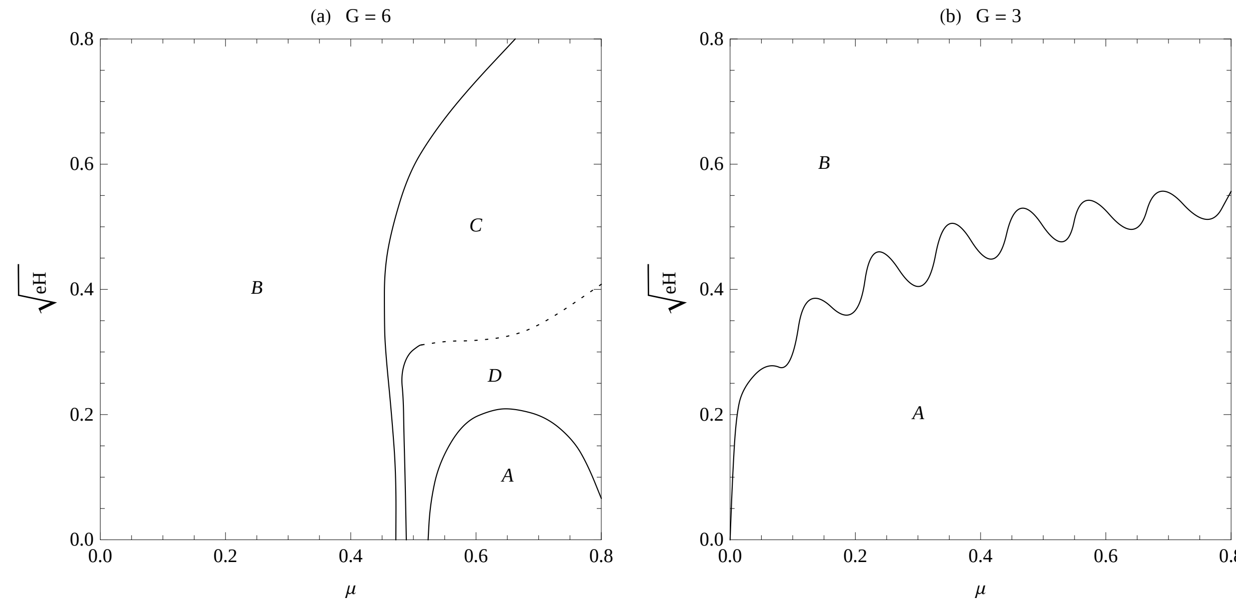

FIG. 1: Phase diagrams for (a) supercritical and (b) subcritical values of the coupling constant G at zero temperature. All quantities are dimensionless. There is a symmetric massless phase A with no chiral condensate and two chirally broken massive phases B and C, the latter being a phase with a nonzero matter density ρ, whereas $\rho = 0$ in phase B. Massive phases B and C are spatially nonuniform when $H > 0$. There is also a new phase D with a strong condensate inhomogeneity (retaining the presence of DCDW in the $H \to 0$ limit). The phase transitions are first order. There is a crossover between phases C and D in a magnetic field strong enough; we have plotted the boundary between them with a dotted line in that region.

V. PHASE DIAGRAM

To construct a phase diagram of the system, we have studied the minima of the regularized thermodynamic potential Ω numerically with respect to the order parameters m and b for different values of the chemical potential μ and the magnetic field strength H. We tried to find a global minimum in the case of several minima present on the Ω surface. We used spectrum (10) and took into account corrections (14) to study the stability of the results with respect to small deviations of $\mathbf{b}$ from the direction of the magnetic field (taking $\Lambda' = 10\,\Lambda$). We only used dimensionless quantities throughout the calculations with Λ being the characteristic energy scale. In what follows, we denote these quantities with the same symbols as the original ones, e.g., m stands for m/Λ, etc. We performed integration over the quantum number n instead of summation when $eH \ll 1$, thus being able to consider the limit $H \to 0$ with no singularities. The estimate of the maximum relative and absolute error was set at the level of 10^{-3} and 10^{-8}, respectively. We take the values of μ and $\sqrt{eH}$ from 0 up to 0.8; it should be noted that there is no physical sense in considering high values of these parameters since 1 is the (dimensionless) regularization constant in our model. The critical value of the coupling constant is $G_c \simeq 3.27$ in our model. If $H = 0$ and $\mu = 0$, spontaneous chiral symmetry breaking only occurs when $G > G_c$; we take this fact as the definition of G_c.

At present, an exact form of the one-particle energy spectrum in the case of $b_\perp > 0$ ($m > 0$) is not known, so that comprehensive analysis of the problem cannot be made. There is no strict guarantee that there are no global minima of the thermodynamic potential somewhere in the region $m > 0$, $b > 0$, $b_\perp > 0$ when $H > 0$, $\mu > 0$ since one can construct a dimensionless ratio $\sqrt{eH}/\mu$ and the latter may be related to the ratio $b_\perp/b$. If this is true, the DCDW wave vector orientation would be diverted from the preferred direction of $\mathbf{H}$ and the rotational symmetry of the system would be completely broken. Nonetheless, it is reasonable to believe that the global minima of Ω are reached when $b_\perp = 0$ implying that the rotational symmetry is still preserved. To test this to the extent possible, for each minimum found (when $H > 0$), we studied the behavior of the thermodynamic potential in the region of $b_\perp$ close to zero. We actually calculated the second derivative $\partial^2\Omega/\partial b_\perp^2|_{b_\perp \to 0}$ numerically, and we made use of the explicit energy spectrum corrections (14) during the evaluation of that quantity. The latter turned to be positive everywhere, so, in this approximation, no instability of the thermodynamic potential minima with respect to $b_\perp$ has been found.

The results of numerical analysis in the case of $T = 0$ and supercritical $G = 6$ are presented in Figs. 1a, 2, 3. As one would expect, we recover the result obtained in Ref. [20] in the limit $H \to 0$, see Fig. 3(a); and there is a nontrivial behavior of the system when $H > 0$. The order parameter b related to the DCDW wave vector ($b = q/2$) grows either smoothly (for the range of the chemical potential μ up to some value) or discontinuously (for higher values of μ) with

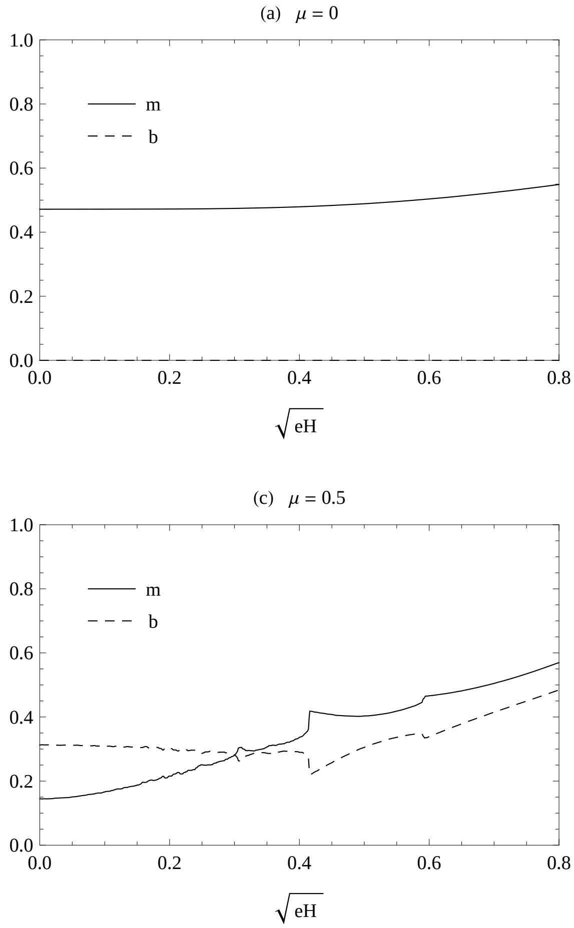

FIG. 2: The order parameters as functions of the magnetic field strength for various values of the chemical potential at zero temperature, and $G = 6$. All quantities are dimensionless.

the increase of the magnetic field strength H, the effect being more vivid for greater μ, see Fig. 2. There is also a gap corresponding to a transition from a symmetric phase present in a weak field in dense matter (we assume $b = 0$ when $m = 0$ although b has no physical meaning in that case and can be set to have an arbitrary value). DCDW is absent if $\mu = 0$. Noticeably, the order parameter b grows linearly with the increase of the chemical potential μ (up to a critical value where a phase transition occurs), the growth rate being higher in the stronger field, see Fig. 3. DCDW is absent when $H = 0$ except for the range of μ corresponding to a new phase examined in Ref. [20]. This phase (we name it phase D, see below) undergoes further development with the increase of the magnetic field strength forcing out the symmetric phase with $m = 0$. The order parameter oscillation visible in the diagrams when the chemical potential is high enough is a phenomenon typical for the model [59–61], such behavior is generally inherent in cold many-body quantum systems in a magnetic field, the fact known since the studies on the de Haas–van Alphen effect [62, 63].

In general, the NJL model is known to give rise to three distinct phases [64, 65]: a symmetric massless phase A with no chiral condensate and two chirally broken massive phases B and C, the latter being a phase with a nonzero matter density ρ, whereas $\rho = 0$ in phase B (the $B \to C$ transition occurs for $\mu > m$). Massive phases B and C are now spatially nonuniform when $H > 0$. There is also a new phase with a strong condensate inhomogeneity (retaining the presence of DCDW in the $H \to 0$ limit studied in Ref. [20]), we denote it with the symbol D. The position of the phases described above in case of $H = 0$ is illustrated in Fig. 3(a) and their evolution with the increase of the magnetic field strength is shown in Fig. 1(a). The transitions between the phases under consideration are first order since the order parameters are discontinuous except for the $B \to C$ transition when $H = 0$ which is second order being a singular point in the diagram [61]. It should be noted that the $D \to C$ transition occurring in a magnetic field strong enough actually belongs to a series of typical order parameter oscillations visible, e.g., in Figs. 2c, 2d. There is no significant physical difference between C and D in that region so we consider it as a crossover area plotting the corresponding transition with a dotted line, and we only plot a solid line between C and D in the region where these phases can be distinguished clearly with a noticeable change in their physical properties [see, e.g., Figs. 3a, 3b]. The

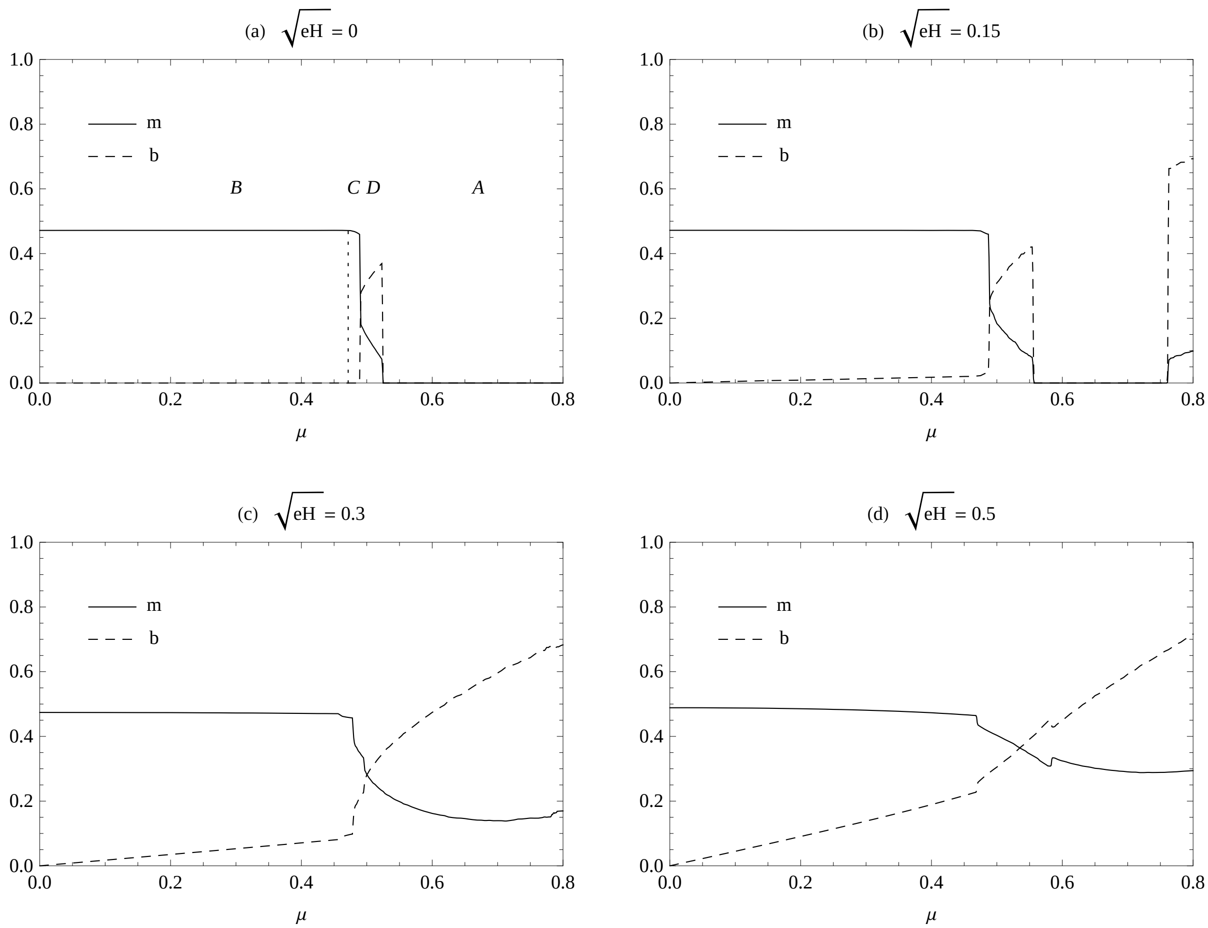

FIG. 3: The order parameters as functions of the chemical potential for various values of the magnetic field strength at zero temperature, and $G = 6$. All quantities are dimensionless.

position of the end point separating the solid and the dotted segment is therefore not fixed precisely and should be chosen judiciously. In the most general case, one may consider an infinite series of phases $\{A_n\}$, $\{C_n\}$, and $\{D_n\}$ when a magnetic field is present with phase transitions corresponding to the order parameter oscillations mentioned above (like it is done in Refs. [59–61], see also a recent study in Ref. [35]). However, since these oscillations are small in their relative magnitude and tend to be smeared out with finite temperature taken into account, we consider such series as single phases, and in this reasonable approximation, this situation may be treated as a crossover between C and D. The main result we have obtained is that there is a nonzero b in all phases when $H > 0$ except for the symmetric one and the case of $\mu = 0$. Smooth and linear growth of b with the increase of H and μ is inherent in phase B. Symmetric phase A now occupies a limited area on the diagram.

We have also examined the case of a subcritical G in addition to the strong-coupling regime. The results for $G = 3$, $T = 0$ are presented in Figs. 1b, 4. The magnetic field is known to be a catalyst of the spontaneous chiral symmetry breaking both in renormalizable and nonrenormalizable (NJL-like) theories (see, e.g., Refs. [66–75] and also Refs. [59–61]), the latter demonstrating the emergence of a dynamic fermion mass for arbitrary small values of the coupling constant. This effect is present in our case as well. The phase diagram structure obtained for our model is similar to that derived in Ref. [61] for $G < G_c$ and phase B exhibits the same behavior of the order parameter b growth as described above. Thus, DCDW formation is preferable for the system in a wide range of the coupling constant.

VI. CONCLUSIONS

The calculations performed in the framework of the NJL model have shown that the presence of an external magnetic field favors the formation of a spatially nonuniform chiral condensate (in the form of a dual chiral density wave) in dense quark matter at low temperatures. This means that there exists a critical magnetic field strength H_c

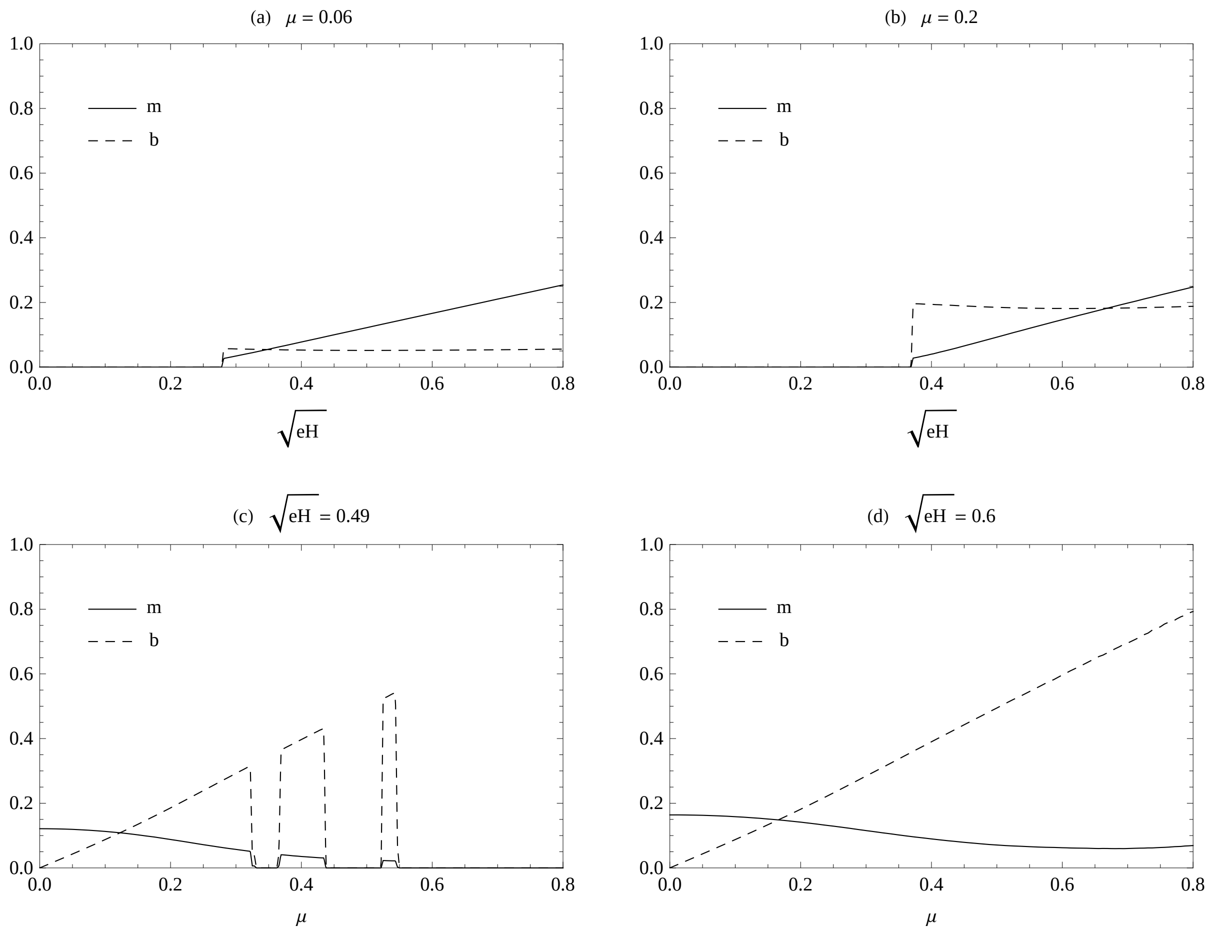

FIG. 4: Samples of the order parameter dependence on the external conditions at zero temperature, $G = 3$. All quantities are dimensionless.

such that one of the spatially nonuniform DCDW phases (B, C, or D) emerges in the system when $H > H_c$ both for supercritical and subcritical values of the coupling constant for arbitrary nonzero values of the chemical potential μ and $T = 0$. For example, if $G > G_c$ then it is easy to see that $H_c > 0$ for the range of μ corresponding to the symmetric phase A, whereas $H_c = 0$ for other values of μ [see Fig. 1(a)]. On the contrary, in the case of subcritical $G < G_c$, the quantity H_c is nonvanishing for all $\mu > 0$ [see Fig. 1(b)]. One can verify that the effect of the chiral condensate spatial modulation is mainly due to the particle and antiparticle energy spectra asymmetry induced by the presence of DCDW in our model; if one drops out the contribution of the distorted lowest Landau level (LLL) to the thermodynamic potential of the system, the phenomenon of the condensate wave vector being nonzero in the massive phases of the model except for D will be lost and phase D will be far less stable occupying a small area on the phase diagram.

As discussed in Ref. [20] (see also Ref. [17]), linear growth of the condensate wave vector with the increase of the chemical potential is generally inherent in one-dimensional systems and this is in agreement with the dimensional reduction phenomenon occurring for fermions in a strong magnetic field [71–73] (see also Ref. [76] for the case of chromomagnetic fields); this behavior of the order parameters is actually related to the specific properties of the LLL. A singular role of the LLL and its impact on physical phenomena in various problems concerning dense matter and symmetry breaking is pointed out in a number of studies, see, e.g., Ref. [53] and also a recent discussion on the chiral magnetic effect in Ref. [77].

In this paper, we have only reported our results for the case of cold quark matter, and the role of finite temperature is to be studied in our forthcoming publications. There are also other interesting subjects left beyond the scope of our paper. Since quark matter may possess its own magnetization (see, e.g., Refs. [78–81] and also Refs. [42, 44]), a challenging self-consistent problem may arise with the magnetic field being generated dynamically. One should also consider the color superconductivity phenomenon possible along with the chiral density waves formation; the results obtained in such generalized models seem to be less regularization dependent [22]. A nonzero quark current mass

should be taken into account as well. Besides, concerning the ground state spatial configuration of the NJL model, it has been argued that domain walls may be more preferable than chiral density waves at least in the absence of external gauge fields [29]. At the same time, as it may be concluded from our calculations, one would expect that a strong magnetic field favors the formation of DCDW. Thus, there should exist some solution interpolating between the two extremes in the intermediate region of the magnetic field strength, possibly being similar to the solution discussed in Ref. [82]. On the other hand, a competing mechanism for domain walls formation in a strong magnetic field has also been discussed in literature [42]. Therefore, the problem of the preferred ground state spatial structure requires further theoretical investigation but, in general, it has been shown that an external magnetic field induces condensate inhomogeneity, in the form of DCDW or some more preferable configuration. Another subject of research is obtaining analytical expressions for the order parameters as functions of the external conditions in a weak magnetic field at least in some special cases using an approach similar to that adopted, e.g., in Ref. [83].

Our concluding remark is that real existence of a spatially nonuniform chiral condensate in nature is yet an open question since theoretical results related to the problem are generally model and approximation dependent. Unfortunately, exact QCD calculation of its production is impossible, since this is an infrared phenomenon. Nonetheless, we believe that the theoretical research of this kind of nonperturbative effects will yield our better understanding the properties of strongly interacting matter.

Appendix: Regularization of Ω'_μ at the $n = 0$ Level

Let us consider the contribution of the $n = 0$ energy level in expression (21):

$$K^{\mathrm{en}} = \int_0^{+\infty} dp \sum_\epsilon \left(|E-\mu| - |E|\right)\, \theta(\Lambda' - |E|), \quad E = \epsilon\sqrt{m^2+p^2} + b, \tag{A.1}$$

where we have utilized the parity of E with respect to p under the integral and omitted the constant factor $-eH/(2\pi)^2$. One can easily prove the following formulas being valid for sufficiently large values of P:

$$\begin{aligned}
\int_0^P dp\, \left|\sqrt{m^2+p^2}+a\right| &= \begin{cases} I(0,P) + Pa, & a > -|m|, \\ -I(0,p_0) + I(p_0,P) - 2p_0 a + Pa, & a < -|m|, \end{cases} \\
\int_0^P dp\, \left|-\sqrt{m^2+p^2}+a\right| &= \begin{cases} I(0,P) - Pa, & a < |m|, \\ -I(0,p_0) + I(p_0,P) + 2p_0 a - Pa, & a > |m|, \end{cases}
\end{aligned}$$

where $p_0 = \sqrt{a^2 - m^2}$ and

$$I(p_1,p_2) \equiv \int_{p_1}^{p_2} dp\, \sqrt{m^2+p^2} = \frac{p_2}{2}\sqrt{m^2+p_2^2} - \frac{p_1}{2}\sqrt{m^2+p_1^2} + \frac{m^2}{2}\ln\left(\frac{p_2+\sqrt{m^2+p_2^2}}{p_1+\sqrt{m^2+p_1^2}}\right).$$

Let us apply the above formulas to evaluate K^{en}. We have to choose different integration limits for different values of ϵ due to the energy cutoff and the spectrum asymmetry: $P_1 = \sqrt{(\Lambda'-b)^2 - m^2}$ for $\epsilon = +1$ and $P_2 = \sqrt{(\Lambda'+b)^2 - m^2}$ for $\epsilon = -1$. Consider the following expression:

$$\begin{aligned}
J^{\mathrm{en}} &= \int_0^{P_1} dp\, \left|\sqrt{m^2+p^2}+a\right| + \int_0^{P_2} dp\, \left|-\sqrt{m^2+p^2}+a\right| \\
&= \begin{cases} I(p_0,P_1) + I(p_0,P_2) + (P_1-P_2)a + 2p_0 a, & |m| < a, \\ I(0,P_1) + I(0,P_2) + (P_1-P_2)a, & -|m| < a < |m|, \\ I(p_0,P_1) + I(p_0,P_2) + (P_1-P_2)a - 2p_0 a, & a < -|m|. \end{cases}
\end{aligned} \tag{A.2}$$

Let us compare this result to one obtained with the help of a trivial momentum cutoff (i.e., without the factor $\theta(\Lambda' - |E|)$ but with a common upper limit in the integrals). Assuming $P_1 = P_2 = \tilde{\Lambda}'$, we get

$$\begin{aligned} J^{\mathrm{mom}} &= \int\limits_0^{\tilde{\Lambda}'} dp \left|\sqrt{m^2+p^2}+a\right| + \int\limits_0^{\tilde{\Lambda}'} dp \left|-\sqrt{m^2+p^2}+a\right| \\ &= \begin{cases} 2I(p_0,\tilde{\Lambda}') + 2p_0 a, & |m| < a, \\ 2I(0,\tilde{\Lambda}'), & -|m| < a < |m|, \\ 2I(p_0,\tilde{\Lambda}') - 2p_0 a, & a < -|m|. \end{cases} \end{aligned} \tag{A.3}$$

The difference of the above quantities is

$$J^{\mathrm{en}} - J^{\mathrm{mom}} = I(\tilde{\Lambda}', P_1) + I(\tilde{\Lambda}', P_2) + (P_1 - P_2)a.$$

Thus we have

$$K^{\mathrm{en}} = J^{\mathrm{en}}|_{a=b-\mu} - J^{\mathrm{en}}|_{a=b} = J^{\mathrm{mom}}|_{a=b-\mu} - J^{\mathrm{mom}}|_{a=b} + \mu(P_2 - P_1) = K^{\mathrm{mom}} + \mu(P_2 - P_1),$$

where we have introduced the symbol K^{mom} to denote expression (A.1) regularized with a trivial momentum cutoff instead of the $\theta(\Lambda' - |E|)$ factor:

$$K^{\mathrm{mom}} = \int\limits_0^{\tilde{\Lambda}'} dp \sum_{\epsilon} \left(|E-\mu| - |E|\right). \tag{A.4}$$

Since

$$\lim_{\Lambda'\to\infty} (P_2 - P_1) = 2b,$$

we finally get

$$\int\limits_0^{+\infty} dp \sum_{\epsilon} \left(|E-\mu| - |E|\right) \theta(\Lambda' - |E|)\Bigg|_{\Lambda'\to\infty} = \int\limits_0^{\tilde{\Lambda}'} dp \sum_{\epsilon} \left(|E-\mu| - |E|\right)\Bigg|_{\tilde{\Lambda}'\to\infty} + 2\mu b. \tag{A.5}$$

We should note that expression (A.4) for K^{mom} is convergent and well-defined in the limit $\tilde{\Lambda}' \to \infty$. Taking sufficiently large values of p, one has

$$\sum_{\epsilon} \left(\left|\epsilon\sqrt{m^2+p^2} + b - \mu\right| - \left|\epsilon\sqrt{m^2+p^2} + b\right| \right) = 0.$$

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